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Path Integral Formulation for Chern-Simons Quantum Mechanics

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Abstract Chern-Simons quantum mechanics is studied by using the path integral formulation. We find that the model can be decomposed into two independent oscillators when a set of new coordinate is chosen. The propagator is constructed in this new set of coordinate and the spectra is read off directly from it. The spectra will be divergent when the mass of the particle tends to zero. In order to get a physical result, one must regularize the spectra properly. We afford a new scheme of regularization. Interestingly, our scheme shows that the regularization we proposed amounts to erase one of the oscillators in the new set of coordinate. Physical interpretations for the regularization are given and some ambiguities in the literatures are clarified.

Keywords Chern-Simons quantum mechanics · Path integral · Constraints

Some years ago, Chern-Simons quantum mechanics was studied in [1]. It describes a charged particle confined by a quadratic scalar potential in the constant external magnetic field. It is found that the reduced model, which is obtained by taking the limit of the mass of the particle tending to zero from the full model, behaves interestingly. The authors studied the models from both classical and quantum aspects. They showed that the model had the continuous limit in its classical aspect. However, the quantum aspect, say, the spectra, do not have such continuous limit. They got the spectra by employing the operator form and found that the spectra of the full model would be divergent when the mass tended to zero.

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In order to make the spectra of the full theory match that of the reduced theory when the mass tends to zero, one must subtract an infinity part together with a finite part from the spectra of the full theory artificially. However, the interpretations of this subtraction are not convincing. Recently, a different conclusion was announced in [2]. Basing on the operator ordering theories, the author found that contrary to previous conclusion [1], one only need subtract an infinite part from the spectra. It is quite interesting that they got different results by using the same formulation of quantum mechanics (operator formulation).

The above authors got different results based on the same form of the quantum mechanics, it is necessary to study this problem from another method.

The other form of quantum mechanics, say, the path integral formulation, may avoid the problem of operator ordering since all the numbers in the path integral formulation are C-number. In this letter, we shall employ this formulation to analyze the Chern-Simons quantum mechanics. We find that we can decompose the model into two orthogonal independent oscillators. Interestingly enough, the frequency of one oscillator will keep finite while the other will tend to infinity when the mass takes zero value. According to this observation, we propose a physical scheme of regularization and a physical interpretation.

The action of the Chern-Simons quantum mechanics is [1]:

$$L = \frac{M_0}{2}\dot{x}_i^2 + \frac{B}{2}\epsilon^{ij}x_i\dot{x}_j - \frac{K}{2}x_i^2, \tag{1}$$

in which $i, j = 1, 2$. It describes a charged ($q = 1$) particle with mass M_0 confined by a quadratic potential moves in the plane with a perpendicular magnetic field.

The Lagrangian (1) can also be written in the first order form. It is:

$$L = (p_i - A_i)\dot{x}^i - H. \tag{2}$$

Where A_i are magnetic potentials. They can be written as

$$A_i = \frac{B}{2}\epsilon_{ij}x^j, \tag{3}$$

when the symmetrical gauge is chosen. H is Hamiltonian:

$$H = \frac{p_i^2}{2M_0} + \frac{K}{2}x_i^2. \tag{4}$$

Using the symmetrical gauge, we can write the Lagrangian (1) as

$$L = p_i\dot{x}^i - \frac{B}{2}\epsilon_{ij}x^j\dot{x}^i - \frac{K}{2}x_i^2. \tag{5}$$

Considering the Lagrangian (2) and the expression of the magnetic potential (3), we can get the following classical Poisson brackets

$$\{x^i, x^j\} = 0, \qquad \{x^i, p_j\} = \delta_j^i, \qquad \{p_i, p_j\} = B\epsilon_{ij} \tag{6}$$

by using Faddeev-Jackiw method [3].

The reduced theory, which is the limit of the mass tending to zero from the full theory, can be written in the first-order form as:

$$L = \frac{B}{2}\epsilon^{ij}x_i\dot{x}_j - \frac{K}{2}x_i^2. \tag{7}$$

It is found in [1] that the classical aspect, say, the solutions of classical equations of motion, of these two models match each other naively when the mass of the particle tends to zero. In terms of the operator form, the authors found that the energy spectra of the reduced theory (2) can not match that of the full theory (1) trivially when the same limit is taken. In order to made them coincide with each other, the authors subtracted an infinite part together with a finite part from the spectra of the full theory artificially. However, the physical interpretations are not convincing. And recently, a different subtraction scheme was reported in [2]. In that paper the author also started from the operator form but he found that only a infinite part was necessary to be subtracted.

In this paper, we shall study this problem by using path integral formulation. One of the advantages of path integral formulation is that one do not need care about the problem of operator ordering problem since all the numbers in path integral formulation are C-number. As a result, the problem of operator ordering can be avoided. Furthermore, our work clarify the ambiguities in the present literature.

The key point of the path integral formulation is to construct propagator [4–6]. It can be defined either in the coordinate space or in the momentum space in the ordinary quantum mechanics. We can define the propagator in coordinate space as $K(\vec{x}_b, t_b|\vec{x}_a, t_a)$. However, we would like to define the propagator as (the advantages of defining propagator in this way will be obvious shortly after)

$$\begin{aligned} K(x_b^1, p_{2b}, t_b|x_a^1, p_{2a}, t_a) &\equiv \langle x_b^1, p_{2b}, t_b|x_a^1, p_{2a}, t_a\rangle \\ &= \langle x_b^1, p_{2b}|\exp\left\{-i\frac{H}{\hbar}T\right\}|x_a^1, p_{2a}\rangle, \end{aligned} \tag{8}$$

where $T = t_b - t_a$.

The propagator (8) can be written as [4–6]

$$K(x_b^1, p_{2b}, t_b|x_a^1, p_{2a}, t_a) = \prod_{k=1,2}\int \mathcal{D}x^k\mathcal{D}p_k \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, L\right\}, \tag{9}$$

where L is the Lagrangian (1).

We denote $|n\rangle$ as the eigenstates of energy whose corresponding eigenvalues are E_n. They satisfy:

$$\sum_n |n\rangle\langle n| = 1. \tag{10}$$

Sandwiching (10) into (8), we are led to

$$\begin{aligned} &K(x_b^1, p_{2b}, t_b|x_a^1, p_{2a}, t_a) \\ &\quad= \sum_n \exp\left\{-i\frac{E_n}{\hbar}T\right\}\varphi_n^*(x_b^1, p_{2b})\varphi_n(x_a^1, p_{2a}), \end{aligned} \tag{11}$$

where $\varphi_n(x^1, p_2) = \langle x^1, p_2|n\rangle$ are the eigenfunctions expressed in terms of variables x^1 and p_2. It means that we can read off both eigenvalues and eigenfunctions if the propagator (9) is constructed.

Because the wave functions depend only on x^1 and p_2, we must get rid of the variables x^2 and p_1. It can be achieved by integrating with respect to the variables x^2 and p_1 directly

since the integrand (9) is an exponential of a quadratic form of them. After some algebraic calculations, we get the following result:

$$K(x_b^1, p_{2b}, t_b | x_a^1, p_{2a}, t_a) = N \int \mathcal{D}x^1 \mathcal{D}p_2 \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L_{eff}\right\}, \tag{12}$$

where N is a numerical factor and L_{eff} is the Lagrangian expressed in terms of the variables x^1 and p_2,

$$L_{eff} = \frac{B^2 + M_0 K}{2K}\dot{x}_1^2 + \frac{B}{K}\dot{x}_1\dot{p}_2 + \frac{1}{2K}\dot{p}_2^2 - \left(\frac{p_2^2}{2M_0} + \frac{K}{2}x_1^2\right). \tag{13}$$

The standard method to calculate the propagator (12) is to write $x^1 = x_{cl}^1 + \delta x^1$ and $p_2 = p_{2cl} + \delta p_2$ (where x_{cl}^1 and p_{2cl} are solutions of classical equations of motion which are obtained by minimizing the action (13)) and then substitute them into the propagator (12). The propagator can then be written as [4–6]:

$$K(x_b^1, p_{2b}, t_b | x_a^1, p_{2a}, t_a) = N F(t_b - t_a) \exp\left\{\frac{i}{\hbar} S_{eff}^{cl}\right\}. \tag{14}$$

In which $F(t_b - t_a)$ is the quantum fluctuation and $S_{eff}^{cl} = \int_{t_a}^{t_b} dt L_{eff}(x_{cl}^1, \dot{x}_{cl}^1, p_{2cl}, \dot{p}_{2cl})$ is the action along the classical trajectory. However, since there is a mixed term $\frac{B}{K}\dot{x}^1\dot{p}_2$ in the Lagrangian (13), the process of direct calculation will be tedious.

We can simplify the process by eliminating the mixed term. This method was proposed recently by us during the study of the noncommutative harmonic oscillators [7]. We denote $\mathbf{X} = (x^1, p_2)^T$. Then the Lagrangian (13) can be written in a compact form as:

$$L_{eff} = \frac{1}{2}\dot{\mathbf{X}}^T\mathbf{M}\dot{\mathbf{X}} - \frac{1}{2}\mathbf{X}^T\mathbf{N}\mathbf{X}, \tag{15}$$

in which $\mathbf{M}$ and $\mathbf{N}$ are two 2×2 matrices,

$$\mathbf{M} = \begin{pmatrix} \frac{B^2+M_0K}{K} & \frac{B}{K} \\ \frac{B}{K} & \frac{1}{K} \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} K & 0 \\ 0 & \frac{1}{M_0} \end{pmatrix}. \tag{16}$$

In order to eliminate the mixed term in the above Lagrangian, we perform a scalar transformation firstly and then perform an orthogonal rotation. The transformation for $\mathbf{X}$ is: $\mathbf{X} \to \mathbf{Y} = \mathbf{OSX}$, where $\mathbf{Y} = (y_1, y_2)^T$. Correspondingly, in order to keep the Lagrangian (13) invariant, the matrices $\mathbf{M}$ and $\mathbf{N}$ should also be transformed accordingly. The transformations for them are: $\mathbf{M} \to \mathbf{OS^{-1}M(OS)}^{-1}$ and $\mathbf{N} \to \mathbf{OS^{-1}N(OS)}^{-1}$. In which $\mathbf{S}$ is a scalar transformation matrix and $\mathbf{O}$ is an orthogonal matrix. The explicit expressions are,

$$\mathbf{S} = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \sqrt{\frac{1}{M_0}} \end{pmatrix}, \qquad \mathbf{O} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned}\cos\alpha &= -\frac{\mathcal{B}}{\sqrt{\mathcal{B}^2+4(\mathcal{A}-\lambda_1)^2}} = -\frac{2(\mathcal{A}-\lambda_2)}{\sqrt{\mathcal{B}^2+4(\mathcal{A}-\lambda_2)^2}},\\ \sin\alpha &= -\frac{\mathcal{B}}{\sqrt{\mathcal{B}^2+4(\mathcal{A}-\lambda_2)^2}} = \frac{2(\mathcal{A}-\lambda_1)}{\sqrt{\mathcal{B}^2+4(\mathcal{A}-\lambda_1)^2}},\end{aligned} \tag{18}$$

in which

$$\mathcal{A} = \frac{B^2+M_0K}{K^2}, \qquad \mathcal{B} = \frac{2B}{K}\sqrt{\frac{M_0}{K}}, \qquad \mathcal{C} = \frac{M_0}{K}, \tag{19}$$

and

$$\begin{aligned}\lambda_1 &= \frac{1}{2}[\mathcal{A}+\mathcal{C}+\sqrt{\mathcal{B}^2+(\mathcal{A}-\mathcal{C})^2}]\\ &= \frac{1}{2K^2}[2M_0K+B^2+B\sqrt{4M_0K+B^2}],\\ \lambda_2 &= \frac{1}{2}[\mathcal{A}+\mathcal{C}-\sqrt{\mathcal{B}^2+(\mathcal{A}-\mathcal{C})^2}]\\ &= \frac{1}{2K^2}[2M_0K+B^2-B\sqrt{4M_0K+B^2}].\end{aligned} \tag{20}$$

After the transformation, the effective Lagrangian (13) is written as:

$$L_{eff} = \frac{1}{2}M_i\dot{y}_i^2 - \frac{1}{2}M_i\Omega_i^2y_i^2, \quad i=1,2, \tag{21}$$

where

$$M_i = \lambda_i \quad \text{and} \quad \Omega_i^2 = \frac{1}{\lambda_i}. \tag{22}$$

Obviously, the Lagrangian (21) describes two orthogonal independent harmonic oscillators with masses M_i and frequencies Ω_i, respectively. It shows that we have decomposed the model (1) or (2) into two independent oscillators in the new coordinate y_i. The propagator (12) can be calculated in the standard way. It is [4–6]

$$\begin{aligned}K(\vec{Y}_b,t_b|\vec{Y}_a,t_a) \sim &\sqrt{\prod_{k=1}^{2}\frac{M_k\Omega_k}{2\pi i\hbar\sin\Omega_kT}}\\ &\times\exp\left\{\sum_{k=1}^{2}\frac{iM_k\Omega_k}{2\hbar\sin\Omega_kT}[(y_{kb}^2+y_{ka}^2)\cos\Omega_kT-2y_{ka}y_{kb}]\right\}.\end{aligned} \tag{23}$$

The eigenvalues are very familiar. They are:

$$E_{n_1,n_2} = \hbar\Omega_1\left(n_1+\frac{1}{2}\right)+\hbar\Omega_2\left(n_2+\frac{1}{2}\right), \quad n_1,n_2=0,1,2,\ldots \tag{24}$$

where

$$\Omega_1 = \frac{-B + \sqrt{4M_0K + B^2}}{2M_0},$$
$$\Omega_2 = \frac{B + \sqrt{4M_0K + B^2}}{2M_0}. \tag{25}$$

The eigenfunctions can also be read off directly from the propagator (23). They are

$$\phi(y_1, y_2) \sim \prod_{i=1}^{2} \exp\left[-\frac{1}{2}\alpha_i y_i\right] H_{n_k}(\alpha_i y_i) \tag{26}$$

in which the H_n are the Hermite polynomials and the α_i are dimensionless parameters defined as

$$\alpha_i = \sqrt{\frac{M_i \Omega_i}{\hbar}}. \tag{27}$$

Our results (24) agree with the results obtained in [1]. To view it, we notice that the eigenvalues obtained in [1] are:

$$E_{n_\rho, m} = \hbar\omega(2n_\rho + |m| + 1) - m\hbar\omega_L, \quad n_\rho = 0, 1, 2, \ldots, m = 0, \pm 1, \pm 2, \ldots \tag{28}$$

where

$$\omega^2 = \omega_L^2 + \frac{K}{M_0}, \qquad \omega_L = \frac{B}{2M_0}. \tag{29}$$

The eigenvalues (28) can be written in the form of two orthogonal harmonic oscillators with frequencies ω_1 and ω_2 respectively:

$$E_{n_1, n_2} = \hbar\omega_1\left(n_1 + \frac{1}{2}\right) + \hbar\omega_2\left(n_2 + \frac{1}{2}\right), \quad n_1, n_2 = 0, 1, 2, \ldots \tag{30}$$

in which

$$\omega_1 = -\omega_L + \sqrt{\frac{K}{M_0} + \omega_L^2}, \qquad \omega_2 = \omega_L + \sqrt{\frac{K}{M_0} + \omega_L^2} \tag{31}$$

and

$$n_\rho = n_2 + \frac{m - |m|}{2}, \quad m = n_1 - n_2. \tag{32}$$

Clearly, the results (24) are equal to that of (28). It shows that the results we obtained using the path integral formulation are equivalent to the previous ones (28) which were obtained from the operator form.

Now we consider the limit of $M_0 \to 0$. A simple calculation shows that

$$\lim_{M_0 \to 0} \Omega_1 = \frac{K}{B},$$
$$\lim_{M_0 \to 0} \Omega_2 = \frac{B}{M_0} + \frac{K}{B} \to \infty. \tag{33}$$

Evidently, one of the frequencies will tend to infinity when the mass tends to zero. In the other words, the spectra will be divergent in this limit. In order to get a physical result, a regularization procedure is mandatory. That is, we must get rid of the infinity by hand. Here, we make the regularization as:

$$\Omega_2 = 0. \tag{34}$$

It means that in addition to an infinity, a finite part, say $\frac{K}{B}$, also needs to be deleted. The same situation is also found in the [1]. However, in that paper, the authors have not afforded a lucid physical interpretation. Here, we afford an intuitive, but a physical explanation of the truncation (34) as follows: the Ω_2 tends to infinity when the mass tends to zero. It means that both the energy of the ground state and the energy intervals are infinity. Therefore, this degree of freedom, say, y_2, can not be occupied at all. We can decouple this degree of freedom safely. So we conclude that the spectra of the full theory in the limit of $M_0 \to 0$ are:

$$E_n = \hbar \frac{K}{B}\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{35}$$

The corresponding eigenfunctions are very familiar ones. They are:

$$\phi(x^1) \sim \exp\left[-\frac{1}{2}\alpha_1 y_1\right] H_n(\alpha_1 y_1) \tag{36}$$

where α_1 has been given in (27).

In fact, the necessities of truncation can also be reflected directly from the propagator (23). When $M \to 0$, we can directly find that:

$$\lim_{M_0 \to 0} M_2 \Omega_2 = \sqrt{\lambda}_2 \to 0. \tag{37}$$

It shows that the propagator is suppressed in the directly of y_2. As a result, there is only one direction left when m takes its zero value. Our results show that we indeed need to subtract a finite part together with an infinite part.

In order to verify the truncation (34) we made is correct, let us investigate the reduced theory (7) which is obtained from the Lagrangian (1) by taking the limit of $M_0 \to 0$. It is well known that reduction of phase space will change the symplectic structure. Since the Lagrangian (7) is the first-order, it is convenient to read commutator between x^i using Faddeev-Jackiw method [3]. Since the reduced Lagrangian (7) is already the first-form, we can read the canonical variables $\xi^i = (x^1, x^2)$ and the corresponding canonical one-forms $a^i = (-\frac{B}{2}x^2, \frac{B}{2}x^1)$ from it. The symplectic matrix is defined as $f_{ij} = \frac{\partial a^j}{\partial \xi^i} - \frac{\partial a^i}{\partial \xi^j}$. It is

$$f = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}. \tag{38}$$

Obviously, the symplectic matrix is not singular and the inverse matrix is

$$f^{-1} = \begin{pmatrix} 0 & -\frac{1}{B} \\ \frac{1}{B} & 0 \end{pmatrix}. \tag{39}$$

The commutative relations between variables x^i can be read directly from f^{-1}. They are

$$\{x^i, x^j\} = -\frac{1}{B}\epsilon^{ij}. \tag{40}$$

The Hamiltonian of the reduced theory can also been read from the first-order Lagrangian (7). It is

$$H_R = \frac{K}{2}(x^i)^2. \tag{41}$$

Considering the commutative relation (40) and the reduced Hamiltonian (41), we can write down the eigenvalues directly. They are nothing but (35).

In fact, the above conclusion can also be achieved from Dirac's theory [8]. Noticing that in order to keep the Hamiltonian (4) to be finite when mass tends to zero, we have no other choices but to set $p_i = 0$, $i = 1, 2$. It meas that we have two primary constraints

$$\varphi_i = p_i \approx 0, \quad i = 1, 2 \tag{42}$$

in the terminology of Dirac [8].

Since the commutator between the constraints $\varphi_i \approx 0$ is no-vanishing (6), they belong to the second class. And it is straightforward to verify by using Dirac process [8] that there are no secondary constraints. The commutators (Dirac brackets) among the remained variables x^i can be computed easily according to the definition. They are

$$\begin{aligned}\{x^i, x^j\}_D &= \{x^i, x^j\} - \{x^i, \varphi_m\}\{\varphi_m, \varphi_n\}^{-1}\{\varphi_n, x^j\} \\ &= -\frac{1}{B}\epsilon^{ij}.\end{aligned} \tag{43}$$

It is nothing but the (40) we obtained from Faddeev-Jackiw method before. The spectra can be obtained by considering the reduced Hamiltonian (41) and the commutator (43). It is right the expression (35).

In conclusion, we employ the path integral formulation to study the Chern-Simons quantum mechanics in this letter. We find that the full model can be decomposed into two independent harmonic oscillators in a new set of coordinate. The propagator can be calculated conveniently in this new set of coordinate. The eigenvalues and eigenfunctions can be read off directly from it. Interestingly, one of the frequencies, say, Ω_2 will be divergent in the limit of $M_0 \to 0$. In order to get a physical result in this limit, we must regularize the spectra. Based on the observation that the zero point energy and the energy intervals will be infinite along one of the degrees of freedom when the mass tends to zero. We propose a consistent scheme for regularization. This regularization scheme amounts to decouple one of the degrees of freedom when the mass tends to zero since this degree of freedom is frozen completely in this limit.

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